Generalized index theorem for topological superconductors with Yang-Mills-Higgs couplings

Takanori Fujiwara and Takahiro Fukui

Department of Physics, Ibaraki University, Mito 310-8512, Japan

Abstract

We investigate an index theorem for a Bogoliubov-de Gennes Hamiltonian (BdGH) describing a topological superconductor with Yang-Mills-Higgs couplings in arbitrary dimensions. We find that the index of the BdGH is determined solely by the asymptotic behavior of the Higgs fields and is independent of the gauge fields. It can be nonvanishing if the dimensionality of the order parameter space is equal to the spatial dimensions. In the presence of point defects there appear localized zero energy states at the defects. Consistency of the index with the existence of zero energy bound states is examined explicitly in a vortex background in two dimensions and in a monopole background in three dimensions.

1 Introduction

Yang-Mills-Higgs systems admit topologically nontrivial field configurations such as vortices and magnetic monopoles [1, 2]. They are considered to be responsible for various nonperturbative effects in particle physics. Recently topological insulators and superconductors have attracted much interest as a new phase of materials in condensed matter physics. Topological insulators are band insulators whose ground states are characterized by topological numbers [3]. The method of topologically classifying insulating ground states has been generalized to superconducting states described by the mean-field Bogoliubov-de Gennes Hamiltonian (BdGH). This provides a unified framework of the classification of non-interacting fermion systems with respect to time reversal and particle-hole symmetries [4, 5, 6, 7]. The topological classification was further extended to systems with topological defects [8], which enables us to classify zero modes along line defects and zero energy bound states localized at point defects from the point of view of the topological universality class.

For a superconductor with time-reversal symmetry (class BDI), there exists a unitary operator that anticommutes with the BdGH. It defines an extended chiral symmetry. Generically, chiral symmetry ensures the topological stability of the zero energy states, since these states are controlled by the index theorem [9, 10, 11] for the BdGH, implying the robustness under continuous deformations of the order parameters. Remarkably, the extended chiral symmetry can be defined in arbitrary dimensions. This is a sharp contrast to the usual chiral invariance in massless Dirac theories, which only exists in even dimensions. The extended chiral invariance also incorporates internal spin symmetry. By gauging the internal spin symmetry we can reformulate the system as a Yang-Mills-Higgs system, where the space varying gap parameters can be regarded as Higgs fields.

In this paper we investigate the index theorem of the Dirac-type BdGH corresponding to the Yang-Mills-Higgs systems in arbitrary dimensions with special emphasis on the extended chiral invariance. Previously, zero energy bound states in two and three dimensions were investigated by Jackiw and Rebbi [12], and Jackiw and Rossi [13]. The index theoretical approaches of their models were explored by Callias [9] and Weinberg [10]. In particular, the Weinberg index theorem turned out to be quite useful for much more complicated systems such as non-Abelian vortices in a color superconductor [14]. We show that for more generic models with Yang-Mills-Higgs couplings in arbitrary dimensions the Weinberg index theorem is applicable, and the index of the BdGH can be nonvanishing, implying that there appear topological zero energy bound states localized at the point defect if the dimensions of order parameter space is equal to the spatial dimensions. We also see that the index is solely determined by the behaviors of the Higgs fields at the spatial infinities. This is expected in odd dimensions [9], since we have no topological invariant defined by the gauge field. In even dimensions, however, we have two topological invariants, one defined by the Higgs fields and one by the Yang-Mills fields. The expression of the topological invariants seems

to contain both of them. One might consider that the number of zero modes would be affected by including the Yang-Mills fields. We show that this is not the case. These two topological invariants are so combined that a unique index is obtained. This was noted for vortex background in two dimensions [10]. We show this happens in arbitrary dimensions.

This paper is organized as follows. In Sect. 2 we introduce BdGH in arbitrary dimensions and their generalized chiral symmetry. We then define the generalized index for the BdGH. A computation of the topological index is given in Sect. 3. The chiral current at the spatial infinities is investigated in Sect. 4. General properties of the chiral current and the relation with the topological index in even dimensions are given in Sect. 5. Zero modes for vortex and monopole configurations in two and three dimensions are investigated in Sect. 6. Finally, Sect. 7. is devoted to summary and discussions.

2 Extended chiral symmetry of BdGH

We begin with a d dimensional fermion system coupled to an O(d) Higgs field ϕ_a ($a = 1, \cdots, d$). We assume that the system is described by the following Hamiltonian

$$H_0 \;=\; -i\gamma^j\partial_j + \phi(x), \qquad (\phi(x) = \Gamma^a\phi_a(x)) \tag{2.1}$$

where γ^j ($j = 1, \cdots, d$) and $\gamma^{d+a} = \Gamma^a$ form a set of $2d$ dimensional γ matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = \delta^{\mu\nu}$. We introduce the $2d$ dimensional chiral matrix by $\gamma_{2d+1} = (-i)^d\gamma^1\cdots\gamma^{2d} = (-i)^d\gamma^1\cdots\gamma^d\Gamma^1\cdots\Gamma^d$. It anti-commutes with both γ^j and Γ^a. The system then possesses the chiral symmetry in the sense that

$$\{\gamma_{2d+1}, H_0\} \;=\; 0. \tag{2.2}$$

Note that the chiral symmetry can be defined in any dimensions. This is contrasted with the usual chiral symmetry which is only defined in even dimensions. The Hamiltonian (2.1) concerns itself with $2d$ dimensions, d spatial and d internal. The chiral invariance is known to be violated in the presence of the chemical potential. In the present work we are concerned with the chiral symmetric case, where the index of the Hamiltonian is well-defined.

In addition to the chiral invariance we can define particle-hole symmetry. Let us introduce the charge conjugation matrix C by

$$C(\gamma^j)^*C^{-1} = \gamma^j, \qquad C(\Gamma^a)^*C^{-1} = -\Gamma^a. \tag{2.3}$$

It is always possible to find C for a given set of γ^μ. Then H_0 satisfies

$$\mathcal{C}H_0\mathcal{C}^{-1} \;=\; -H_0, \tag{2.4}$$

where $\mathcal{C} = CK$ denotes complex conjugation K followed by the multiplication by C. To each particle state ψ of an energy $E > 0$ we can define a state ψ_c with energy $-E$ by

$$\psi_c(x) = \mathcal{C}\psi(x) = C\psi^*(x). \tag{2.5}$$

Eq. (2.4) ensures the particle-hole symmetry of the spectrum of H_0.

Before going into detailed analysis of the index of the Hamiltonian we introduce spin(d) gauge field

$$A_j(x) \;=\; \frac{1}{2}\Sigma^{ab}A_{abj}(x), \tag{2.6}$$

where $\Sigma^{ab} \equiv [\Gamma^a, \Gamma^b]/4$ are the spin(d) generators and A_{abj} are the components of the gauge field satisfying $A^*_{abj} = A_{abj} = -A_{baj}$. The gauge symmetric generalization of H_0 is then given by

$$H \;=\; -i\gamma^j D_j + \phi(x), \tag{2.7}$$

where $D_j = \partial_j + A_j$ is the covariant derivative. For the Higgs field it is given

$$D_j\phi = \Gamma^a D_j\phi_a, \quad D_j\phi_a = \partial_j\phi_a + A_{abj}\phi_b. \tag{2.8}$$

The field strength is also defined as usual

$$F_{ij} \;=\; \frac{1}{2}\Sigma^{ab}F_{abij}, \quad F_{abij} = \partial_i A_{abj} - \partial_j A_{abi} + A_{aci}A_{cbj} - A_{acj}A_{cbi}. \tag{2.9}$$

Since the spin(d) generators commute with γ_{2d+1} and $\mathcal{C}$, the chiral and particle-hole symmetries remain intact by the generalization.

Unlike Nielsen-Olsen vortex and 't Hooft-Polyakov monopole, which will be treated in Sect. 6, we consider the Yang-Mills and Higgs fields as independent classical background. For the present we only assume that the gauge potential approaches to pure gauge and the Higgs field satisfies $\phi^2 \to |\phi_0|^2$ at the spatial infinities, where $|\phi_0|^2$ is a nonvanishing constant. This gives rise to a finite energy gap. Due to the particle-hole symmetry mentioned above, any eigenstate of nonvanishing energy E is necessarily paired with an eigenstate of energy $-E$. These states are also related with the chiral transformation. The zero modes of H, however, can be unpaired. They can be made self-conjugate under the particle-hole symmetry and can be regarded as Majorana states. For a general Yang-Mills-Higgs background it is not possible to find an explicit form of zero mode wave function. In the chirally symmetric case, however, we can investigate the existence or nonexistence of zero modes by computing the index of H defined by

$$\mathrm{index}H \;=\; n_+ - n_-, \tag{2.10}$$

where $n_\pm$ are the numbers of positive and negative γ_{2d+1} chirality zero energy states. The index of H is known to be a topological invariant, *i.e.*, it is invariant under continuous deformations of the gauge and Higgs fields. The index theorem relates the index with a topological invariant of these fields [9, 10, 11].

To establish the index theorem we rewrite Eq.(2.10) as

$$\begin{aligned}\mathrm{index}H &= \lim_{m\to 0} \mathrm{Tr}\gamma_{2d+1}\frac{m^2}{H^2+m^2} \\ &= \lim_{m\to 0}\int d^dx \lim_{y\to x} \mathrm{tr}\gamma_{2d+1}\frac{m^2}{H^2+m^2}\delta^d(x-y), \end{aligned} \tag{2.11}$$

where Tr stands for the integration over the spatial coordinates as well as the trace on the γ matrices. As is well-known, index of the Dirac operator is related with the chiral anomaly of the axial current divergence. To show this we introduce an axial current $J^i(x)$ by

$$J^i(x) = \lim_{\substack{m\to 0\\ M\to\infty}} \lim_{y\to x} \mathrm{tr}\gamma_{2d+1}\gamma^i\left(\frac{1}{iH+m}-\frac{1}{iH+M}\right)\delta^d(x-y), \tag{2.12}$$

where M is a Pauli-Villars mass to regularize the current at short distances. It is straightforward to show that the axial current divergence can be cast into the form

$$\partial_i J^i(x) = -2\lim_{\substack{m\to 0\\ M\to\infty}} \lim_{y\to x} \mathrm{tr}\gamma_{2d+1}\left(\frac{m^2}{H^2+m^2}-\frac{M^2}{H^2+M^2}\right)\delta^d(x-y). \tag{2.13}$$

At this stage we can take the two limits, $m\to 0$ and $M\to\infty$, separately. Eq.(2.11) then can be written as

$$\mathrm{index}H = -\frac{1}{2}\int_{S_\infty^{d-1}} dS_i J^i(x) + c_d, \tag{2.14}$$

where c_d is the topological index defined by

$$c_d = \lim_{M\to\infty} \mathrm{Tr}\gamma_{2d+1}\frac{M^2}{H^2+M^2} \tag{2.15}$$

and S_∞^{d-1} denotes the infinities of the d dimensional euclidean space. In the case of index theorem on compact manifolds without boundaries the contribution from the chiral current on the rhs of (2.14) vanishes and the index coincides with c_d. Since we are working with the Hamiltonian (2.7) defined on a euclidean space, the surface term can be nontrivial [9, 10, 11]. We expect a nonvanishing contribution to index$H$ if $J^i(x)$ is of order $|x|^{-d+1}$ for $|x|\to\infty$. To compute the integral on the rhs of Eq. (2.14) we only need the leading behavior of the chiral current at spatial infinities. This will be done in Sect. 4.

3 Topological index

We have shown that the index can be written as a sum of the topological index (2.15) and the surface integral of the chiral current. The evaluation of the functional trace on the rhs of (2.15) is similar to that of chiral anomalies. In this section we compute c_d.

Eq. (2.15) can be explicitly written as

$$
\begin{aligned}
c_d &= \lim_{M\to\infty}\int d^dx \lim_{y\to x} \mathrm{tr}\gamma_{2d+1}\frac{M^2}{H^2+M^2}\delta^d(x-y) \\
&= \lim_{M\to\infty}\int d^dx \int \frac{d^dk}{(2\pi)^d} e^{-ikx}\mathrm{tr}\gamma_{2d+1}\frac{M^2}{H^2+M^2}e^{ikx}. \qquad (3.1)
\end{aligned}
$$

Using

$$
\begin{aligned}
e^{-ikx}He^{ikx} &= -i\gamma^i(D_j+ik_j)+\phi, \qquad (3.2)\\
e^{-ikx}H^2e^{ikx} &= -(D_j+ik_j)^2-\frac{1}{2}\gamma^i\gamma^jF_{ij}-i\gamma^jD_j\phi+\phi^2, \qquad (3.3)
\end{aligned}
$$

we can compute the rhs of Eq. (3.1) as

$$
\begin{aligned}
c_d &= \lim_{M\to\infty}\int d^dx\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}\gamma_{2d+1}\frac{M^2}{-(D_j+ik_j)^2-\frac{1}{2}\gamma^i\gamma^jF_{ij}-i\gamma^jD_j\phi+\phi^2+M^2} \\
&= \lim_{M\to\infty}M^2\int d^dx\int\frac{d^dk}{(2\pi)^d}\sum_{n=0}^{\infty}\mathrm{tr}\Bigg[\gamma_{2d+1}\frac{1}{-(D_i+ik_i)^2+\phi^2+M^2} \\
&\quad\times\left(\left(i\gamma^iD_i\phi+\frac{1}{2}\gamma^i\gamma^jF_{ij}\right)\frac{1}{-(D_i+ik_i)^2+\phi^2+M^2}\right)^n\Bigg]. \qquad (3.4)
\end{aligned}
$$

Due to the presence of γ_{2d+1} the terms with $n<d/2$ vanish under the trace, whereas the terms with $n>d/2$ do not contribute to the sum in the limit $M^2\to\infty$ as can be easily seen by the scaling argument of the momentum variables $k\to Mk$. This immediately gives $c_d=0$ in odd dimensions.

For $d=2N$ even we obtain

$$
\begin{aligned}
c_d &= \int d^dx\int\frac{d^{2N}k}{(2\pi)^d}\frac{1}{(k^2+1)^{N+1}}\mathrm{tr}\gamma_{2d+1}\left(\frac{1}{2}\gamma^i\gamma^jF_{ij}\right)^N \\
&= \frac{(-1)^N}{(2\pi)^NN!}\int d^dx\epsilon^{i_1\cdots i_d}\mathrm{tr}_\eta F_{i_1i_2}\cdots F_{i_{d-1}i_d}, \qquad (3.5)
\end{aligned}
$$

where $\epsilon^{i_1\cdots i_d}$ is the Levi-Civita symbol in d dimensions. We have also introduced tr_η by

$$
\mathrm{tr}_\eta(\cdots) = \mathrm{tr}(\eta\cdots), \qquad (3.6)
$$

where η is defined by

$$
\eta = \frac{(-1)^{d(d-1)/2}}{2^d}\Gamma^1\cdots\Gamma^d. \qquad (3.7)
$$

The overall normalization is chosen to give

$$
\mathrm{tr}_\eta\Gamma^{a_1}\cdots\Gamma^{a_d}=\epsilon^{a_1\cdots a_d}. \qquad (3.8)
$$

It satisfies anti-cyclic property

$$\mathrm{tr}_\eta \Gamma\Gamma' = -\mathrm{tr}_\eta \Gamma'\Gamma \quad \text{for} \quad \{\eta, \Gamma\} = 0. \tag{3.9}$$

Later we need to compute traces involving ϕ. In even dimensions η anti-commutes with ϕ, whereas it commutes in odd dimensions.

We have seen that only the gauge field contributes to the topological index and the Higgs field is irrelevant to the computation of c_d. The result (3.5) coincides with computation of index of a Dirac operator without Higgs fields.

4 Computation of chiral current

The chiral current defined by (2.12) is written by the regularized fermion propagator. It is in general a nonlocal quantity of the background fields. To find its contribution to the index of the BdGH we only need the asymptotic behaviors at spatial infinities, where the chiral current turns out to become local. In this section we evaluate the asymptotic form of the chiral current.

We first rewrite the current (2.12) as

$$J^i(x) \;=\; -i \lim_{\substack{m\to 0\\ M\to\infty}} \int \frac{d^dk}{(2\pi)^d} e^{-ikx} \mathrm{tr}\gamma_{2d+1}\gamma^i H \left(\frac{1}{H^2+m^2} - \frac{1}{H^2+M^2} \right) e^{ikx}. \tag{4.1}$$

The computation of $J^i(x)$ is similar to the one presented in Sect. 3. We obtain

$$\begin{aligned} J^i(x) \;&=\; -i \lim_{\substack{m\to 0\\ M\to\infty}} \int \frac{d^dk}{(2\pi)^d} \sum_{n=0}^{\infty} \left\{ \frac{1}{(k^2+|\phi_0|^2+m^2)^{n+1}} - (m^2 \to M^2) \right\} \\ &\quad \times \mathrm{tr}\gamma_{2d+1}\gamma^i(\gamma^j k_j - i\gamma^j D_j + \phi) \left(i\gamma^k D_k\phi + \frac{1}{2}\gamma^k\gamma^l F_{kl} + \Delta \right)^n, \end{aligned} \tag{4.2}$$

where Δ is given by

$$\Delta \;=\; 2ik_iD_i + D_i^2 - |\phi|^2 + |\phi_0|^2. \tag{4.3}$$

At the spatial infinities ϕ^2 approaches to a constant $|\phi_0|^2$. We further assume

$$\phi^2 - |\phi_0|^2,\ \partial_j\phi,\ A_j \;\sim\; \mathrm{O}(|x|^{-1}) \quad \text{for} \quad |x| \to \infty. \tag{4.4}$$

To find index$H$ it is only necessary to know the leading $\mathrm{O}(|x|^{-d+1})$ terms of the current for $|x| \to \infty$. It is easy to convince oneself that the terms of the rhs of Eq. (4.2) vanish for $n < (d-1)/2$ because of the trace with γ_{2n+1}, whereas the terms with $n \geq d$ can be ignored since they decay faster than $|x|^{-d+1}$ for $|x| \to \infty$. Eq. (4.2) then can be written as

$$\begin{aligned} J^i(x) \;&=\; -i \lim_{\substack{m\to 0\\ M\to\infty}} \int \frac{d^dk}{(2\pi)^d} \sum_{n\geq (d-1)/2}^{d-1} \left\{ \frac{1}{(k^2+|\phi_0|^2+m^2)^{n+1}} - (m^2 \to M^2) \right\} \\ &\quad \times \mathrm{tr}\gamma_{2d+1}\gamma^i\phi \left(i\gamma^i D_i\phi + \frac{1}{2}\gamma^i\gamma^j F_{ij} \right)^n + \mathrm{O}(|x|^{-d}). \end{aligned} \tag{4.5}$$

The k integral can be done for $n \geq (d-1)/2$ as

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+\mu^2)^{n+1}} = \frac{\Gamma(n+1-d/2)}{(4\pi)^{d/2} n!} \frac{1}{\mu^{2(n+1)-d}}. \tag{4.6}$$

Now the limits $m \to 0$ and $M \to \infty$ can be taken safely. Keeping only the nonvanishing contributions, we obtain

$$J^i(x) = -i \sum_{l=0}^{\left[\frac{d-1}{2}\right]} \frac{\Gamma(d/2-l)}{(4\pi)^{d/2}(d-l-1)!} \mathrm{tr}\gamma_{2d+1}\gamma^i \hat{\phi} \mathrm{Symm}\left[\left(i\gamma^j D_j \hat{\phi}\right)^{d-2l-1} \left(\frac{1}{2}\gamma^j\gamma^k F_{jk}\right)^l\right] + \mathrm{O}(|x|^{-d}), \tag{4.7}$$

where $\hat{\phi} = \phi/|\phi_0|$ and Symm denotes symmetrized product defined by

$$\mathrm{Symm} A^n B^m = \frac{(n+m)!}{n!m!} \frac{\partial^{n+m}}{\partial s^n \partial t^m} \exp[sA+tB]\bigg|_{s=t=0}. \tag{4.8}$$

Since $\gamma^j D_j \hat{\phi}$ and $\gamma^j\gamma^k F_{jk}$ are effectively commutative in the trace of Eq. (4.7), we can simplify the symmetrized product further. We thus arrive at the asymptotic form of the chiral current

$$J^i(x) = \frac{(-1)^{d-1}}{(4\pi)^{d/2}} \sum_{k=0}^{\left[\frac{d-1}{2}\right]} \frac{2^{d-k}\Gamma(d/2-k)}{k!(d-2k-1)!} \epsilon^{ii_2\cdots i_d} \mathrm{tr}_\eta \hat{\phi} D_{i_2}\hat{\phi} \cdots D_{i_{d-2k}} \hat{\phi} F_{i_{d-2k+1} i_{d-2k+2}} \cdots F_{i_{d-1} i_d}, \tag{4.9}$$

where nonleading contributions are suppressed. The asymptotic chiral current is local with respect to the background fields. Unlike the topological index c_d it exists in odd as well as even dimensions.

It is easy to obtain explicit forms of $J^i(x)$ in low dimensions. In two dimensions the chiral current and topological index are given by

$$J^i(x) = \frac{1}{\pi}\epsilon^{ij}\epsilon^{ab}\hat{\phi}_a \partial_j \hat{\phi}_b - \frac{1}{2\pi}\epsilon^{ij}\epsilon^{ab} A_{abj}, \tag{4.10}$$

$$c_2 = -\frac{1}{4\pi}\int d^2x \epsilon^{ij}\epsilon^{ab}\partial_i A_{abj}. \tag{4.11}$$

These lead to the index

$$\mathrm{index} H = \frac{1}{2\pi}\int_{S^1_\infty} dS_i \epsilon^{ij}\epsilon^{ab}\hat{\phi}_a \partial_j \hat{\phi}_b. \tag{4.12}$$

We see that the topological index is canceled by the gauge field dependent term of the chiral current [10].

Similar thing also happens in three dimensions. The topological index vanish identically in odd dimensions and the gauge field dependent terms of the chiral current can be cast into a total derivative term as

$$\begin{aligned} J^i(x) &= -\frac{1}{4\pi}\epsilon^{ijk}\epsilon^{abc}\left(\hat\phi_a D_j\hat\phi_b D_k\hat\phi_c + \frac{1}{2}\hat\phi_a F_{jkbc}\right) \\ &= -\frac{1}{4\pi}\epsilon^{ijk}\epsilon^{abc}\hat\phi_a\partial_j\hat\phi_b\partial_k\hat\phi_c - \frac{1}{4\pi}\epsilon^{ijk}\epsilon^{abc}\partial_j(A_{abk}\hat\phi_c). \end{aligned} \tag{4.13}$$

We thus obtain the index

$$\mathrm{index}H = \frac{1}{8\pi}\int_{S^2_\infty} dS_i\epsilon^{ijk}\epsilon^{abc}\hat\phi_a\partial_j\hat\phi_b\partial_k\hat\phi_c. \tag{4.14}$$

Again the gauge field dependence disappears and the index is only determined by the asymptotic behavior of the Higgs field at the infinities [9]. In the next section we show that this holds true in arbitrary dimensions.

5 Differential geometric approach

To establish the gauge field independence of index$H$ it is convenient to introduce Lie algebra valued differential forms [16], the gauge potential 1-form and the field strength 2-form, as

$$A = \frac{1}{4}\Gamma^a\Gamma^b \mathrm{d}x^i A_{abi}, \tag{5.1}$$

$$F = \frac{1}{2}\mathrm{d}x^i\mathrm{d}x^j F_{ij} = \frac{1}{8}\mathrm{d}x^i\mathrm{d}x^j\Gamma^a\Gamma^b F_{abij} = \mathrm{d}A + A^2. \tag{5.2}$$

Note that F satisfies Bianchi identity

$$DF = dF + AF - FA = 0. \tag{5.3}$$

The exterior covariant derivative of $\hat\phi$ is given by

$$D\hat\phi = \mathrm{d}\hat\phi + [A, \hat\phi]. \tag{5.4}$$

Taking D once more we obtain

$$D^2\hat\phi = [F, \hat\phi]. \tag{5.5}$$

Since the generators of spin(d) commutes with η defined by (3.7), so do A and F in arbitrary dimensions.

The topological index (3.5) in $d = 2N$ dimensions can be written in terms of F as

$$c_d = \frac{(-1)^N}{\pi^N N!}\int \mathrm{tr}_\eta F^N, \tag{5.6}$$

where the integral is taken over the entire d dimensional space. If we introduce Chern-Simons form ω^0_{d-1} by

$$\mathrm{tr}_\eta F^N = \mathrm{d}\omega^0_{d-1}, \tag{5.7}$$

the topological index (5.6) can be converted to the surface integral

$$c_d = \frac{(-1)^N}{\pi^N N!}\int_{S^{d-1}_\infty}\omega^0_{d-1}, \tag{5.8}$$

where S^{d-1}_∞ denotes the $(d-1)$-sphere at the spatial infinities. The Chern-Simons form ω^0_{d-1} can be written as

$$\omega^0_{d-1} = N\int_0^1 dt \mathrm{tr}_\eta A(t\mathrm{d}A + t^2A^2)^{N-1}. \tag{5.9}$$

For $d = 2, 4, 6$ the Chern-Simons forms are explicitly given by

$$\begin{aligned}
&\omega^0_1 = \mathrm{tr}_\eta A,\\
&\omega^0_3 = \mathrm{tr}_\eta\left(A\mathrm{d}A + \frac{2}{3}A^3\right),\\
&\omega^0_5 = \mathrm{tr}_\eta\left(A(\mathrm{d}A)^2 + \frac{3}{2}A^3\mathrm{d}A + \frac{3}{5}A^5\right).
\end{aligned} \tag{5.10}$$

We now turn to the chiral current (4.9). The first integral in the rhs of Eq. (2.14) can be written as

$$\int_{S^{d-1}_\infty} dS_i J^i = \int_{S^{d-1}_\infty} {}^*J, \tag{5.11}$$

where ${}^*J$ is the Hodge dual of the chiral current 1-form $J = J_i(x)\mathrm{d}x^i$ and is given by

$${}^*J = \frac{1}{(d-1)!}\epsilon_{i_1 i_2\cdots i_d}J^{i_1}(x)\mathrm{d}x^{i_2}\cdots\mathrm{d}x^{i_d}. \tag{5.12}$$

For the current Eq. (4.9), the dual can be written as

$${}^*J = \sum_{k=0}^{\left[\frac{d-1}{2}\right]} C_{d,k}\mathrm{tr}_\eta\hat{\phi}(D\hat{\phi})^{d-2k-1}F^k, \tag{5.13}$$

where $C_{d,k}$ is defined by

$$C_{d,k} = -(-1)^{\left[\frac{d+1}{2}\right]}\frac{\Gamma(d/2-k)}{\pi^{d/2}k!(d-2k-1)!}. \tag{5.14}$$

Taking the exterior derivative of the current and using Eq. (5.3) and (5.5), we get

$$\begin{aligned}
\mathrm{d}^*J &= \sum_{k=0}^{\left[\frac{d-1}{2}\right]} C_{d,k}\mathrm{tr}_\eta((D\hat{\phi})^{d-2k} + \hat{\phi}[F,\hat{\phi}](D\hat{\phi})^{d-2k-2} - \hat{\phi}D\hat{\phi}[F,\hat{\phi}](D\hat{\phi})^{d-2k-3} \\
&\quad + \cdots + (-1)^{d-2k-2}\hat{\phi}(D\hat{\phi})^{d-2k-2}[F,\hat{\phi}])F^k \\
&= \sum_{k=0}^{\left[\frac{d-1}{2}\right]} C_{d,k}\mathrm{tr}_\eta(D\hat{\phi})^{d-2k}F^k - \sum_{k=0}^{\left[\frac{d-1}{2}\right]}(d-2k-1)C_{d,k}\mathrm{tr}_\eta[F,\hat{\phi}]\hat{\phi}(D\hat{\phi})^{d-2k-2}F^k.
\end{aligned} \tag{5.15}$$

In deriving this use has been made of the fact that $[F,\hat{\phi}]$ effectively anti-commutes with $\hat{\phi}$ and $D\hat{\phi}$ in the trace. We can simplify the trace in the second summand on the rhs of Eq. (5.15) as

$$\begin{aligned}
\mathrm{tr}_\eta[F,\hat{\phi}]\hat{\phi}(D\hat{\phi})^{d-2k-2}F^k &= -\frac{1}{k+1}\mathrm{tr}_\eta\hat{\phi}(D\hat{\phi})^{d-2k-2}([F,\hat{\phi}]F^k + F[F,\hat{\phi}]F^{k-1}\cdots + F^k[F,\hat{\phi}]) \\
&= -\frac{1}{k+1}\mathrm{tr}_\eta\hat{\phi}(D\hat{\phi})^{d-2k-2}[F^{k+1},\hat{\phi}] \\
&= (-1)^d\frac{2}{k+1}\mathrm{tr}_\eta(D\hat{\phi})^{d-2k-2}F^{k+1}.
\end{aligned} \tag{5.16}$$

Inserting this into Eq. (5.15), we obtain

$$\mathrm{d}^*J = -\frac{2(d-2k+1)}{k}C_{d,k-1}\mathrm{tr}_\eta(D\hat{\phi})^{d-2k}F^k\Big|_{k=\left[\frac{d-1}{2}\right]+1}, \tag{5.17}$$

where use has been made of $\mathrm{tr}_\eta(D\hat{\phi})^d = 0$ and the relation

$$C_{d,k} = \frac{2(d-2k+1)}{k}C_{d,k-1}, \qquad \left(k = 1,2,\cdots,\left[\frac{d-1}{2}\right]\right). \tag{5.18}$$

The former can be verified by noting $\hat{\phi}^2 = 1$.

In odd dimensions the coefficient on the rhs of (5.17) vanishes. We therefore obtain

$$\mathrm{d}^*J = 0. \tag{5.19}$$

This implies that the current can be written as

$$-\frac{1}{2}{}^*J = \frac{(-1)^{\frac{d+1}{2}}}{2^d\pi^{\frac{d-1}{2}}(\frac{d-1}{2})!}\mathrm{tr}_\eta\hat{\phi}(\mathrm{d}\hat{\phi})^{d-1} + \mathrm{d}\Omega_{d-2}(\hat{\phi},A), \tag{5.20}$$

where Ω_{d-2} is a $(d-2)$-form.

In even $(d=2N)$ dimensions Eq. (5.17) can be written as

$$\mathrm{d}^*J = 2\frac{(-1)^N}{\pi^N N!}\mathrm{tr}_\eta F^N. \tag{5.21}$$

This is reminiscent of chiral anomaly. It also implies that ${}^*J$ can be written as

$$-\frac{1}{2}{}^*J = (-1)^{\frac{d}{2}}\frac{\left(\frac{d}{2}\right)!}{\pi^{\frac{d}{2}}d!}\mathrm{tr}_\eta\hat{\phi}(\mathrm{d}\hat{\phi})^{d-1}+\mathrm{d}\Omega_{d-2}(\hat{\phi},A)-\frac{(-1)^{\frac{d}{2}}}{\pi^{\frac{d}{2}}\left(\frac{d}{2}\right)!}\omega^0_{d-1}. \qquad (5.22)$$

Interestingly enough, there appears the the Chern-Simons form.

It is straightforward to check Eq. (5.20) and (5.22) in low dimensions. We give explicit expressions for $d = 2,\ 3,\ 4,\ 5$:

$$d=2: \quad {}^*J=\frac{1}{\pi}\mathrm{tr}_\eta\hat{\phi}\mathrm{d}\hat{\phi}-\frac{1}{\pi}\mathrm{tr}_\eta A,$$

$$d=3: \quad {}^*J=-\frac{1}{4\pi}\mathrm{tr}_\eta\hat{\phi}(\mathrm{d}\hat{\phi})^2-\frac{1}{\pi}\mathrm{d}\mathrm{tr}_\eta\hat{\phi}A,$$

$$d=4: \quad {}^*J=-\frac{1}{6\pi^2}\mathrm{tr}_\eta\hat{\phi}(\mathrm{d}\hat{\phi})^3+\frac{1}{\pi^2}\mathrm{d}\mathrm{tr}_\eta\left(\hat{\phi}\mathrm{d}\hat{\phi}A+\frac{1}{2}\hat{\phi}A\hat{\phi}A\right)+\frac{1}{\pi^2}\mathrm{tr}_\eta\left(A\mathrm{d}A+\frac{2}{3}A^3\right),$$

$$d=5:$$

$${}^*J=\frac{1}{32\pi^2}\mathrm{tr}_\eta\hat{\phi}(\mathrm{d}\hat{\phi})^4+\frac{1}{4\pi^2}\mathrm{d}\mathrm{tr}_\eta\left\{\hat{\phi}(\mathrm{d}\hat{\phi})^2A+\hat{\phi}\mathrm{d}\hat{\phi}A\hat{\phi}A+\frac{1}{3}(\hat{\phi}A)^3+\hat{\phi}(A\mathrm{d}A+\mathrm{d}AA+A^3)\right\}. \qquad (5.23)$$

We now turn to index$H$. We see from Eq. (5.20) or (5.22) that Ω_{d-2} does not contribute to the index and the topological index in even dimensions is canceled by the Chern-Simons form. We thus obtain

$$\mathrm{index}H = (-1)^{\left[\frac{d+1}{2}\right]}\frac{\Gamma(d/2+1)}{\pi^{d/2}d!}\int_{S^{d-1}_\infty}\mathrm{tr}_\eta\hat{\phi}(\mathrm{d}\hat{\phi})^{d-1}. \qquad (5.24)$$

The gauge fields disappear completely and the index is determined only by the behaviors of the Higgs fields at the spatial infinities. This generalizes the observation of Ref. [10] for the two dimensional model of Jackiw and Rossi [13]. In the next section we will introduce Yukawa coupling constant g by substituting ϕ by $g\phi$ in Eq. (2.7). In even dimensions the index (5.24) is not affected by this, whereas an extra overall factor $\mathrm{sgn}(g)$ appears on the rhs of Eq. (5.24) in odd dimensions.

6 Topological configurations in two and three dimensions

So far we have considered the Higgs and gauge fields as independent background fields. In some model field theories topological objects can be realized dynamically as a solution to the field equations. More specifically we assume the gauge-Higgs system in d spatial dimensions with a static energy

$$\mathcal{H} = \int d^dx\left(\frac{1}{8e^2}(F_{abij})^2+\frac{1}{2}(D_i\phi_a)^2+\frac{\lambda_0}{8}(|\phi_0|^2-|\phi|^2)^2\right), \qquad (6.1)$$

where e is the gauge coupling constant. It becomes stationary for the fields satisfying

$$D_i D_i \hat{\phi}_a + \lambda(1 - |\hat{\phi}|^2)\hat{\phi}_a = 0, \qquad D_i F_{abij} + \kappa \hat{\phi}_a \overleftrightarrow{D}_j \hat{\phi}_b = 0, \tag{6.2}$$

where $\kappa = e^2|\phi_0|^2$ and $\lambda = \lambda_0|\phi_0|^2/2$. These nonlinear field equations give rise to topological objects, Nielsen-Olesen vortex in two dimensions [1] and 't Hooft-Polyakov monopole in three dimensions [2]. We consider the BdG equation with these topological configurations and see more closely how the index relation is fulfilled.

6.1 Vortex in Maxwell-Higgs system

In two dimensions Eq. (6.2) can describe vortices. Ansatz for a vortex with unit vorticity is given by

$$\hat{\phi}_a(x) = h(r)\frac{x^a}{r}, \qquad A_{abi}(x) = -\epsilon_{ab}\epsilon_{ij}(1 - k(r))\frac{x^j}{r^2}, \tag{6.3}$$

where $h(r)$ and $k(r)$ are assumed to satisfy the following boundary conditions

$$h(0) = 0, \quad k(0) = 1, \quad h(\infty) = 1, \quad k(\infty) = 0. \tag{6.4}$$

Eqs. (6.2) give differential equations for $h(r)$ and $k(r)$ as

$$h'' + \frac{h'}{r} = \frac{k^2 h}{r^2} - \lambda(1 - h^2)h, \qquad k'' - \frac{k'}{r} = \kappa h^2 k. \tag{6.5}$$

We see that $1 - h$ and k decrease exponentially for large r.

We now turn to index$H$ for the vortex background. The covariant derivative $D_i\hat{\phi}_a$ is given by

$$D_i\hat{\phi}_a = h'\frac{x^i x^a}{r^2} + \epsilon_{ij}\epsilon_{ab} hk\frac{x^j x^b}{r^3}. \tag{6.6}$$

It decays exponentially at $r \to \infty$, so does the chiral current $*J$. We see that index$H$ coincides with the topological index c_2 since the chiral current has no contribution to the index. It is easy to compute c_2. The field strength can be found as

$$F_{abij} = -\epsilon_{ab}\epsilon_{ij}\frac{k'}{r}. \tag{6.7}$$

Eq. (5.8) then immediately gives index$H = c_2 = -1$.

This might be felt contradictive with the result of Sect. 5 that index$H$ is determined completely by the asymptotic behavior of the Higgs field. It is of course not the case. The gauge field is related with the Higgs field by the field equations. In particular the chiral current vanishes exponentially as $r \to \infty$. Therefore, the contribution from the Higgs current, the first term on the rhs of (4.10), cancels that from the gauge current, the second

term of the same equation, which in turn exactly cancels the topological index of the gauge field strength (6.7). This implies that the topological invariant (4.12) coincides with the topological index.

The nonvanishing index suggests that the Hamiltonian given by Eq. (2.7) has one negative chirality zero mode. For the vortex background we can find zero mode of H explicitly. To see this we employ the following set of γ matrices

$$\gamma^j = \sigma^j \otimes \sigma^1, \qquad \Gamma^1 = \sigma^3 \otimes \sigma^1, \qquad \Gamma^2 = 1 \otimes \sigma^2. \qquad (j = 1, 2) \tag{6.8}$$

The spin(2) generator and chiral matrix is given by

$$\Sigma_3 = \Sigma^{12} = \frac{i}{2}\sigma^3 \otimes \sigma^3, \qquad \gamma_5 = (-i)^2 \gamma^1 \gamma^2 \Gamma^1 \Gamma^2 = 1 \otimes \sigma^3. \tag{6.9}$$

Zero mode wave function can be chosen to be chiral. We write it in chiral spinors $\psi_\pm$ as

$$\psi_\pm(x) \;\; = \;\; \begin{pmatrix} u_\pm(x) \\ v_\pm(x) \end{pmatrix} \otimes \chi_\pm, \tag{6.10}$$

where $\chi_\pm$ are two component eigenspinors with $\sigma^3 \chi_\pm = \pm \chi_\pm$. Each component of the chiral zero mode satisfies in polar coordinates

$$\begin{aligned} \left\{ -ie^{i\theta} \left(\partial_r + \frac{i}{r}\partial_\theta \right) \pm i \frac{1-k}{r} e^{i\theta} \right\} u_\pm - g|\phi_0| h e^{\mp i\theta} v_\pm &= 0, \\ g|\phi_0| h e^{\pm i\theta} u_\pm + \left\{ -ie^{-i\theta} \left(\partial_r - \frac{i}{r}\partial_\theta \right) \pm i \frac{1-k}{r} e^{-i\theta} \right\} v_\pm &= 0, \end{aligned} \tag{6.11}$$

where we have introduced Yukawa coupling constant g by replacing ϕ with $g\phi$ in Eq. (2.7). The index (4.12) is not affected by this change as mentioned in Sect. 5.

For the negative chirality zero mode we can assume that u_- and v_- are independent of θ. It is easy to check that a normalizable solution is given by

$$u_- = -i\mathrm{sgn}(g) v_- = C_0 \exp\left[-\int_0^r \left(|g\phi_0| h(r') + \frac{1-k(r')}{r'} \right) dr' \right], \tag{6.12}$$

where C_0 is a normalization constant. In Figure 1 we give a plot for the profile of $u_-(r)$ together with $h(r)$ and $k(r)$. The zero mode wave function is localized around the vortex core.

As for the positive chirality zero mode, we can separate the angle variable by assuming

$$u_+(x) = f_m(r) e^{im\theta}, \qquad v_+(x) = i g_m(r) e^{i(m+2)\theta}, \tag{6.13}$$

where m is an integer. f_m and g_m satisfy

$$\begin{aligned} &\left(\frac{d}{dr} - \frac{m}{r} - \frac{1-k}{r} \right) f_m + g|\phi_0| h g_m = 0, \\ &\left(\frac{d}{dr} + \frac{m+2}{r} - \frac{1-k}{r} \right) g_m + g|\phi_0| h f_m = 0. \end{aligned} \tag{6.14}$$

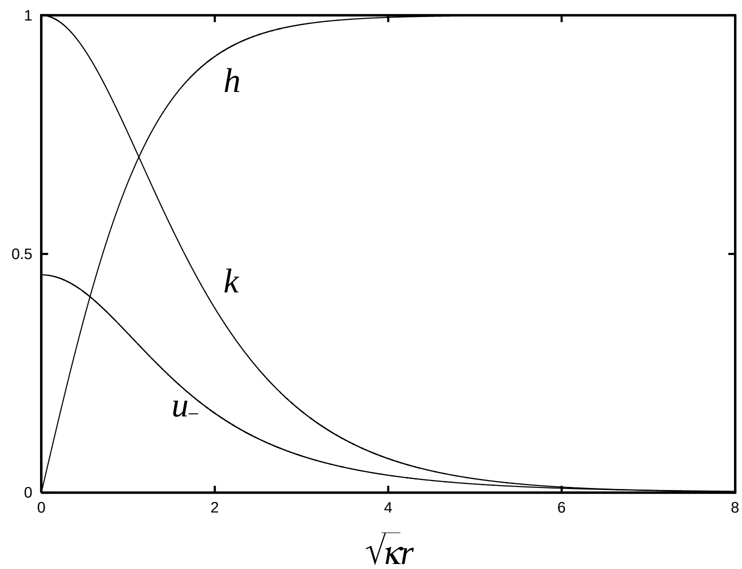

Figure 1: Profiles of $h(r)$, $k(r)$ and $u_-(r)$ with $\lambda = \kappa$ and $g = e/2$.

These lead to the behaviors $f_m \sim r^m$ and $g_m \sim r^{-m-2}$ as $r \to 0$. We see that only one of the two independent solution is regular at the origin. Such a regular solution, however, contains exponentially growing component $\sim e^{|\phi_0| r}$ as $r \to \infty$. We thus conclude that there is no positive chirality zero mode.

6.2 't Hooft-Polyakov monopole

Next we consider a Yang-Mills-Higgs system with spin(3) gauge symmetry in three spatial dimensions. The ansatz for the monopole of unit magnetic charge is given by

$$\hat{\phi}_a(x) = h(r)\frac{x^a}{r}, \qquad A_{abi}(x) = -(1-k(r))\frac{\delta_{ia}x^b - \delta_{ib}x^a}{r^2}. \tag{6.15}$$

Eqs. (6.2) are satisfied if $h(r)$ and $k(r)$ obey the following differential equations

$$h'' + \frac{2}{r}h' = \frac{2}{r^2}k^2 h - \lambda(1-h^2)h, \qquad k'' = \kappa h^2 k - \frac{1}{r^2}(1-k^2)k. \tag{6.16}$$

The boundary conditions for $h(r)$ and $k(r)$ take the same form as Eq. (6.4) for the vortex. From Eqs. (6.16) we see the asymptotic behavior $k \sim e^{-\sqrt{\kappa} r}$ and $1-h \sim e^{-\sqrt{2\lambda} r}$ for sufficiently large r. No analytic solution is not known for Eqs. (6.16). See Ref. [15] for a recent high precision numerical study.

It is straightforward to evaluate index$H$. In odd dimensions only chiral current Eq. (4.13) contributes to the index. Note that the covariant derivatives $D_i\hat{\phi}$ decays exponentially as $r \to \infty$ and the field strength approaches

$$F_{abij} \to \frac{\delta_{ia}\delta_{jb}}{r^2} - \frac{\delta_{ia}x^j x^b - \delta_{ib}x^j x^a}{r^4} - (i \leftrightarrow j). \tag{6.17}$$

Keeping terms that survive at $r \to \infty$, we obtain

$$J^i \approx -\frac{1}{8\pi}\mathrm{sgn}(g)\epsilon^{ijk}\epsilon^{abc}\hat{\phi}_a F_{jkbc} = -\mathrm{sgn}(g)\frac{x^i}{2\pi r^3}, \tag{6.18}$$

where the overall factor sgn(g) comes from the introduction of the Yukawa coupling constant. In odd dimensions the index depends on the sign of g. This immediately gives index$H$ = sgn(g). As in the vortex case, it is also possible to obtain the same result by computing the topological invariant (4.14).

The index obtaind above implies the existence of a zero mode of chirality sgn(g). For the monopole background Eq. (6.15) it is also possible to find the wave function for the zero mode. We employ the following representation of the γ matrices

$$\gamma^i = \sigma^i \otimes 1 \otimes \sigma^1, \qquad \Gamma^a = 1 \otimes \sigma^a \otimes \sigma^2. \qquad (i, a = 1, 2, 3) \tag{6.19}$$

The spin(3) generators $\Sigma_a = \frac{1}{2}\epsilon_{abc}\Sigma^{ab}$ and the chiral matrix γ_7 are given by

$$\Sigma_a = \frac{i}{2} 1 \otimes \sigma^a \otimes 1, \qquad \gamma_7 = 1 \otimes 1 \otimes \sigma^3. \tag{6.20}$$

Let us denote the zero mode wave function by chiral components as

$$\psi \;=\; \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{6.21}$$

Then $\psi_\pm$ must satisfy

$$\left(-i\sigma^j \otimes 1\partial_j + \frac{1}{2}\sigma^j \otimes \sigma^a A_{aj} \pm ig 1 \otimes \sigma^a \phi_a\right)\psi_\pm \;=\; 0. \tag{6.22}$$

where A_{ai} is defined by $A_{ai} = \frac{1}{2}\epsilon_{abc}A_{bci}$. These can be cast into 2×2 matrix equations by noting $(A \otimes B\psi_\pm)_{\alpha\beta} = A_{\alpha\gamma}B_{\beta\delta}(\psi_\pm)_{\gamma\delta} = (A\psi_\pm B^T)_{\alpha\beta}$. With this notation Eq. (6.22) for the monopole background Eq.(6.15) can be expressed as

$$-i\sigma^j\partial_j\Psi_\pm - \epsilon_{jab}(1-k)\frac{x^b}{2r^2}\sigma^j\Psi_\pm\sigma^a \mp ig|\phi_0|h\frac{x^a}{r}\Psi_\pm\sigma^a \;=\; 0. \tag{6.23}$$

where $\Psi_\pm$ are defined by $\Psi_\pm = i\psi_\pm\sigma^2$. These have spherically symmetric solutions

$$(\Psi_\pm(x))_{\alpha\beta} \;=\; F_\pm(r)\delta_{\alpha\beta}, \tag{6.24}$$

where $F_\pm$ satisfy

$$F'_\pm(r) \;=\; -\left(\pm g|\phi_0|h(r) + \frac{1-k(r)}{r}\right)F_\pm(r). \tag{6.25}$$

For $g > 0$ the negative chiral component F_- must vanish, otherwise it grows exponentially as $r \to \infty$. We thus arrive at the normalizable positive chiral zero mode

$$\psi = \begin{pmatrix} iF_+(r)\sigma^2 \\ 0 \end{pmatrix}, \tag{6.26}$$

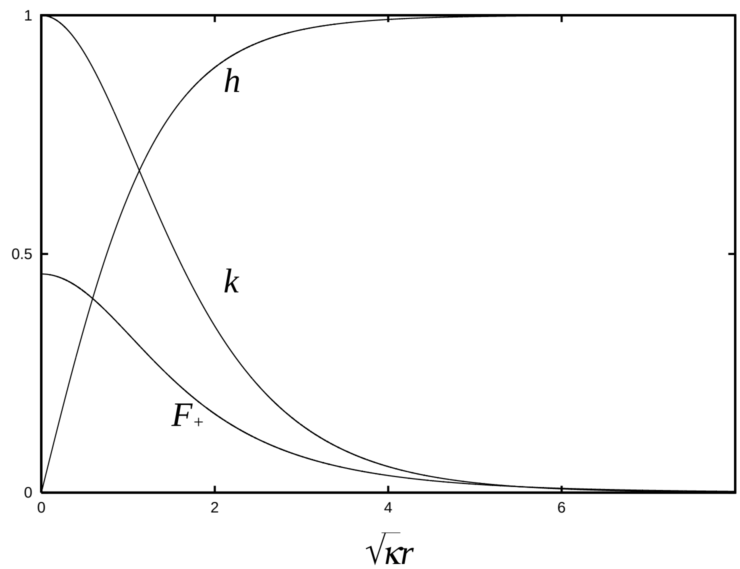

Figure 2: Profiles of $h(r)$, $k(r)$ and $F_+(r)$ for the monopole background with $\lambda = \kappa/2$ and $g = e/2$.

where F_+ is given by

$$F_+ \;=\; C_0 \exp\left[-\int_0^r \left(g|\phi_0|h(r') + \frac{1-k(r')}{r'}\right) dr'\right]. \tag{6.27}$$

Again C_0 is a normalization constant. The zero mode is localized around the monopole and the wave function decays exponentially for $r \to \infty$. In Figure 2 we give a plot of $F_+(r)$ together with $h(r)$ and $k(r)$. The case of $g < 0$ can be analyzed similarly. We obtain one normalizable zero mode with negative chirality. This is consistent with the index theorem.

7 Summary and Discussion

We have evaluated the index of BdGH of a gauged topological insulator or Yang-Mills-Higgs system in arbitrary dimensions by regarding the Higgs and Yang-Mills fields as external backgrounds, which can be set up independently. The index can be expressed as a surface integral of a gauge invariant chiral current plus topological index of the Yang-Mills fields. In odd dimensions the topological index vanishes identically and the gauge field dependent terms of the chiral current can be gathered into a total derivative at spatial infinities, giving no contribution to the index. In even dimensions the gauge field dependent terms of the chiral current can be converted into a total derivative plus the Chern-Simons form, which exactly cancels the topological index. We have thus shown that the index of the BdGH is determined solely by the asymptotic behavior of the Higgs fields whatever topological charge the Yang-Mills field carries.

If the behavior of the Yang-Mills and Higgs fields are governed by some effective Hamiltonian, $D_i\phi_a$ and F_{ij} must decay faster than $|x|^{-d/2}$ for $|x| \to \infty$ to ensure the finiteness of the Hamiltonian. In such systems nonvanishing topological invariant can be obtained only in spatial dimensions less than four. In two dimensions the index of the BdGH is saturated by the topological index. In three dimensions only the ϕF term of the chiral current contributes to the index. This apparently contradicts to the general conclusion that the Yang-Mills field does not contribute to the index. It is , however, possible to have expressions for the index only in terms of ϕ by noting that the gauge fields are related with the Higgs fields by $D_j\hat{\phi}_a = 0$ at the spatial infinities.

We have considered the case of the BdGH containing d order parameters from the beginning. It is possible to consider other systems with less or more order parameters. The evaluation of index of the corresponding BdGH is straightforward. It is rather obvious from our explicit calculations that one cannot obtain nontrivial index unless the number of the order parameters coincides with the spatial dimensions. Our result is consistent with the the topological classification by the Chern number computed from the Berry connection of the Bloch wave functions.

Acknowledgements

This work is supported in part by the Grant-in-Aid for Scientific Research (No. 21540378) from the Japan Society for the Promotion of Science (JSPS) and by the "Topological Quantum Phenomena" Grant-in Aid for Scientific Research on Innovative Areas (No. 23103502) from the Ministry of Education, Culture, Sports, Science and Technology of Japan (MEXT).

References

[1] H.B. Nielsen and P. Olesen, Nucl. Phys. **B61** (1973) 45.

[2] G. 't Hooft, Nucl. Phys. **B79** (1974) 276.

[3] D.J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982);
M. Kohmoto, Ann. Phys. **160** (1985) 343.

[4] M. Zirnbauer, J. Math. Phys. **37** (1996) 4986;

[5] A. Altland and M. Zirnbauer, Phys. Rev. **B55** (1997) 1142.

[6] A. P. Schnyder, S. Ryu, A. Furusaki and A. W. W. Ludwig, Phys. Rev. B **78** (2008) 195125.

[7] A. Kitaev, AIP Conf. Proc. **1134** (2009) 22.

[8] J. C. Teo and C. L. Kane, Phys. Rev. B **78** (2010) 115120.

[9] C. Callias, Commun. Math. Phys. **62**(1978) 213.

[10] E. J. Weinberg, Phys. Rev. **D24** (1981) 2669.

[11] A. J. Niemi and G. W. Semenoff, Phys. Rep. **135** (1986) 99.

[12] R. Jackiw and C. Rebbi, Phys. Rev. **D13** (1976) 3398.

[13] R. Jackiw and P. Rossi, Nucl. Phys. **B190** (1981) 681.

[14] T. Fujiwara, T. Fukui, M. Nitta and S. Yasui, Phys. Rev. **D84** (2011) 076002.

[15] P. Forgács, N. Obadia and S. Reuillon, Phys. Rev. **D71** (2005) 035002.

[16] B. Zumino, Y.-S. Wu and A. Zee, Nucl. Phys. **B239** (1984) 477;
B. Zumino, Lectures given at Les Houches Summer School on Theoretical Physics, Les Houches, France, Aug. 8 - Sept. 2, 1983.